A Katetov-type construction of the Gurarij space

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1 Katetov and the Urysohn space

There is a unique (up to isometry) complete and separable metric space X that satisfies the following property: For any finite metric spaces $N \subseteq M$ and an isometric embedding $f : N \to X$ there is an isometric extension $\overline{f} : M \to X$ [3, 5].

We call this the **Urysohn space** and denote it by $\mathbf{U}$.

In 1986, Katetov presented [3] a construction of the Urysohn space using functions $\varphi : X \to \mathbb{R}$ which represent the one-point extensions of a metric space X . These are the following:

Definition 1. Let X be a metric space. We say that $\xi : X \to \mathbb{R}$ is a **Katetov function** if

$$|\xi(x) - \xi(y)| \le d(x, y) \le \xi(x) + \xi(y).$$

We denote by $K(X)$ the set of Katetov functions in X equipped with the supremum distance.

Proposition 2. *Let (X, d) be a metric space and $x_0 \notin X$. A function $\rho : (X \cup \{x_0\})^2 \to \mathbb{R}$ is a pseudo-metric extension of d if, and only if, $\rho(\cdot, x_0) : X \to \mathbb{R}$ is a Katetov function.*

Lemma 3. *Let X be a metric space. $K(X)$ is a complete metric space and there is an isometric embedding $X \hookrightarrow K(X)$ sending $x \mapsto d(\cdot, x)$.*

Furthermore, this construction is functorial in the sense that an isometric embedding $Y \hookrightarrow X$ gives rise to an isometric embedding $K(Y) \hookrightarrow K(X)$ where $\xi \in K(Y)$ goes to the extension

$$\hat{\xi}(x) = \inf_{y \in Y} \xi(y) + d(x, y). \qquad (1)$$

To ensure separability, we need to work with the completion of the set of Katetov functions with finite support, formally:

$$K_0(X) = \overline{\bigcup \{K(Y) : Y \subseteq X \text{ finite}\}}$$

where $K(Y)$ is seen as a subspace of $K(X)$. As such we still have the isometric inclusion $X \hookrightarrow K_0(X)$ as in Lemma 3. Given any separable metric space X, we may consider an infinite chain of isometric embeddings with $X_0 := X$ and $X_n := K_0(X_{n-1})$, $\forall n \in \mathbb{N}$

$$X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow X_3 \hookrightarrow \ \ldots$$

and by taking the completion of the inductive limit $\overline{\bigcup X_n}$ we have the Urysohn space $\mathbf{U}$.

Proposition 4. *The Urysohn space is a universal polish space. I.e. every polish space embeds isometrically in the Urysohn space.*

2 One-point extensions of normed spaces

The Urysohn space has what is, in many ways, an analog among Banach spaces:

Definition 5. A **Gurarij space** is a separable Banach space $\mathbf{G}$ having the following property: Given any finite dimensional spaces $E \subseteq F$, an isometric embedding $\varphi : E \to \mathbf{G}$, and $\varepsilon > 0$, there exists an ε-isometric extension $\tilde{\varphi} : F \to \mathbf{G}$ of φ. That is,

$$(1 - \varepsilon)\|x\| \le \|\tilde{\varphi}(x)\| \le (1 + \varepsilon)\|x\|.$$

Gurarij proved the existence of this space in 1966 [2] and Lusky proved in 1976 that it is unique up to isometric isomorphisms [4]. We study the construction of the Gurarij space due to Itai Ben Yaacov [1] in a manner analogous to that of Katetov.

Proposition 6. *Let E be a normed space and $x_0 \notin E$. A function $\|\cdot\|' : E \oplus \mathbb{R}x_0 \to \mathbb{R}$ is a semi-norm which extends $\|\cdot\|$ if, and only if, $\|\cdot - x_0\|' : E \to \mathbb{R}$ is a convex Katetov function.*

Definition 7. Let E be a normed space. We denote by $K_C(E)$ the set of convex Katetov functions $\xi : E \to \mathbb{R}$ equipped with the supremum distance.

Analogously to the metric case, we may embed E isometrically in $K_C(E)$ in a functorial manner:

Lemma 8. *Let E be a normed space. The map $E \to K_C(E)$ given by $a \mapsto \|\cdot - a\|$ is an isometric embedding. Moreover, any isometric embedding $E \hookrightarrow F$ induces an isometric embedding $K_C(E) \to K_C(F)$ which sends $\xi \in K_C(E)$ to an extension $\hat{\xi} \in K_C(F)$ as in (1).*

Again, to guarantee separability, we define the subspace

$$K_{C,0}(E) := K_C(E) \cap K_0(E).$$

However, while the space $K_0(X)$ of one-point pseudometric extensions of X is metric, given a normed space E, **the space $K_{C,0}(E)$ of one-point semi-norm extensions isn't a normed space**.

3 Recalling Lipschitz functions

If X is a pointed metric space, the vector space of all Lipschitz functions $f : X \to \mathbb{R}$ where $f(0) = 0$ with a norm given by the Lipschitz constant forms a Banach space denoted by $\mathrm{Lip}_0(X)$.

Lemma 9. *Let X be a pointed metric space. The evaluation map $\delta_X : X \to \mathrm{Lip}_0(X)^*$ given by*

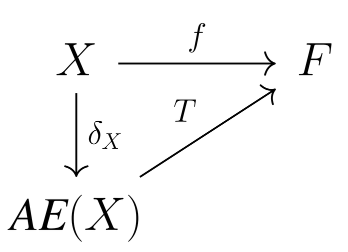
$$\delta_X(x)(f) = f(x) \qquad \forall f \in \mathrm{Lip}_0, \ \forall x \in X,$$

is an isometric embedding which maps $0 \mapsto 0$. Moreover, $\delta_X(X \setminus \{0\})$ is a linearly independent subset of $\mathrm{Lip}_0(X)^*$.

Definition 10. Let X be a pointed metric space. We define the **Arens-Eells space of** X, denoted by $\mathrm{AE}(X)$, as the subspace

$$\mathrm{AE}(X) := \overline{\mathrm{span}}\, \delta_X(X) \subseteq \mathrm{Lip}_0(X)^*.$$

Proposition 11. *Let X be a pointed metric space. $AE(X)^*$ is linearly isometric to $\mathrm{Lip}_0(X)$ and the Arens-Eells space of X is characterized by the following universal property: For any Lipschitz function $f : X \to F$ which preserves the origin, there is a unique $T : AE(X) \to F$ such that the following diagram commutes:*



And $\|f\| = L(f)$.

In this sense, the Arens-Eells space of a given metric space X **gives a Banach space structure to a metric space** in a way that X is identified isometrically in $\mathrm{AE}(X)$ and the bounded linear functionals of said space coincide precisely with the Lipschitz functions over $X \mapsto \mathbb{R}$ which preserve the origin.

4 Metric spaces over E

While it is a good idea to try and use the Arens-Eells space to linearize metric spaces, to deal with the issue of $K_{C,0}(X)$ not having a vector space structure, it is still problematic if we wish to iterate this, since every time we correct the lack of a vector space structure in a new iteration we produce an entirely new vector space without preserving the Banach space structure of the previous step. To work around this, Ben Yaacov employs the following notions:

Definition 12. Let X be a metric space and E a normed space. We say that X is a **metric space over** E if there is an isometric embedding $\varphi : E \to X$ where the map $a \to d(\cdot, \varphi(a))$ is convex for all $a \in E$.

For a normed space E we denote by $\mathrm{Lip}_E(X)$ the space of Lipschitz functions $f : X \to \mathbb{R}$ which are linear on E.

Proposition 13. *Let X be a metric space over a normed space E. There is a Banach space $AE(X, E)$, called the **Arens-Eells space of** X **over** E together with an isometric embedding $X \subseteq AE(X)$ and which is characterized (up to isometric isomorphism) by the following universal property: Every Lipschitz functions $f : X \to F$, where φ is a Banach space, which is linear on E, admits a unique bounded linear extension $\phi : AE(X, E) \to F$ and $\|\phi\| = L(\varphi)$.*

Instead of turning every non-zero element into a linearly independent vector, the Arens-Eells space of X over E gives us a natural way to linearize only the points of a metric space which don't already have some suitable normed space structure.

5 Constructing the Gurarij space

Now, given a Banach space E, we are able to construct an appropriate chain of embeddings of Banach spaces, with $E_0 := E$ and $E_{n+1} := \mathrm{AE}(K_{C,0}(E_n), E_n)$ for all $n \in \mathbb{N}$.

$$E_0 \hookrightarrow E_1 \hookrightarrow E_2 \hookrightarrow E_3 \hookrightarrow \ldots$$

Finally, we arrive at Ben Yaacov's construction:

Theorem 14. *Let E be a separable Banach space. Then the inductive limit E_ω of the chain above is the Gurarij space* $\mathbf{G}$.

Corollary 15. *The Gurarij space is a universal separable normed space. That is, for every separable normed space E there is a linear isometric embedding of E in* $\mathbf{G}$.

6 On polish groups

The motivation for Ben Yaacov's paper was the work of Uspenskij on the isometry group of $\mathbf{U}$ where he proves in a 1990 paper [6] the following:

Theorem 16. *The isometry group of the Urysohn space $\mathrm{Iso}(\mathbf{U})$ is a universal polish group. That is, any polish group embeds there homeomorphically.*

Uspenskij questioned whether the linear isometry group of the Gurarij space is also universal. The answer follows neatly from Ben Yaacov's construction and constitutes the main theorem of his paper:

Theorem 17. *The linear isometry group $\mathrm{Iso}_L(\mathbf{G})$ of the Gurarij space is a universal polish group.*

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