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A brief introduction to infinitary combinatorics through the Galvin-Prikry theorem

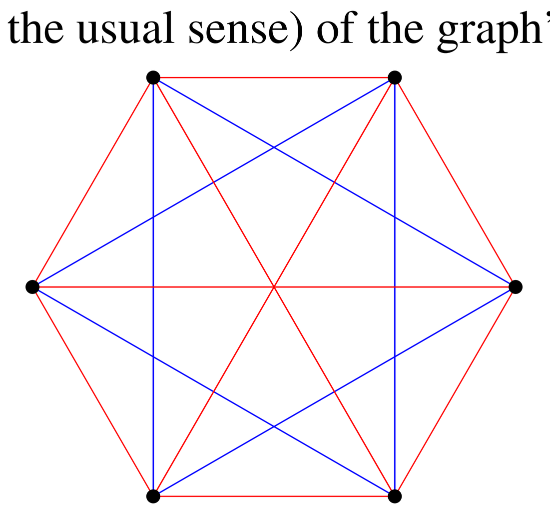
Lucas Nunes Fernandes Teles
Valentin Raphael Henri Ferenczi (supervisor)
Instituto de Matemática e Estatística - USP

1 Ramsey's theorem

The field of combinatorics has Ramsey's theorems as cornerstones. In its essence, these are results about finding a substructure that satisfies some regularity condition when given a large enough structure. This "regularity" is expressed through the concept of colorings: Let A be a set, we define $[A]^n$ to be the set of all n-element subsets of A, for some positive integer n. And a **k-coloring** of $[A]^n$ is a function $c : [A]^n \to \{0, 1, ..., k-1\}$, for $k \geq 1$. Equivalently, a coloring can be seen as a finite partition of $[A]^n$

$$[A]^n = c^{-1}(0) \cup c^{-1}(1) \cup ... \cup c^{-1}(k-1).$$

As an example, $[\{0, 1, 2, 3, 4, 5\}]^2$ can be thought of as a graph and a coloring could be represented as the coloring (in the usual sense) of the graph's edges.



In this example, the "regular substructure" we seek is a subgraph with all it's edges of the same color. Let $B \subseteq A$, we call B **monochromatic** if $[B]^n$ is a subset of $c^{-1}(r)$ for some $r \in \{0, ..., k-1\}$.

Our interest, however, lies in the version of Ramsey's theorem for an infinite set A.

Theorem 1 (Ramsey)**.** *For any countable set A, $n \geq 1$ and any finite k-coloring c of the family $[A]^n$, there is an infinite subset $B \subseteq A$ such that B is monochromatic.*

Many areas of mathematics have been influenced by attempts to find further results of this kind. We focus on how we may extend this theorem to reach the set of the countable subsets of A instead of $[A]^n$.

To simplify our notation we'll work with the set of natural numbers $\mathbb{N}$ instead of an arbitrary countable set A. In this scenario $[\mathbb{N}]^{\aleph_0}$ will represent the set of all countable subsets of $\mathbb{N}$.

Assuming the Axiom of Choice, an improvement of Ramsey's theorem to include the possibility of $[\mathbb{N}]^{\aleph_0}$ is not possible. As we are working in ZFC we omit the necessity of the Axiom of Choice.

Lemma 2. *Given any countable set A and a positive integer k there is a k-coloring such that there is no infinite subset $B \subseteq A$ where $[B]^{\aleph_0}$ is monochromatic.*

An alternative result of this kind comes to us, with stronger requirements than Theorem 1, in the form of the Galvin-Prikry theorem.

2 An advance through topology

To get to the Galvin-Prikry we will work our way towards Ellentuck's theorem, a result with which started the interplay between Ramsey Theory and Topology [1]. As may be expected, we begin by constructing a topological space.

The usual topology of $[\mathbb{N}]^{\aleph_0}$

By defining an onto function $F : [\mathbb{N}]^{\aleph_0} \to 2^{\mathbb{N}}$ where $F(A) = \chi_A$, we relate $[\mathbb{N}]^{\aleph_0}$ to a subset of $2^{\mathbb{N}}$, the space of functions $\mathbb{N} \to 2$. Assuming that $2 = \{0, 1\}$ has the discrete topology we have the product topology in $2^{\mathbb{N}}$ as a natural byproduct. This topology then induces a topology in $F([\mathbb{N}]^{\aleph_0})$ and taking the preimage of these sets we have a topology in $[\mathbb{N}]^{\aleph_0}$. We will call this the **usual topology of** $[\mathbb{N}]^{\aleph_0}$ and denote this topological space by $([\mathbb{N}]^{\aleph_0}, \tau)$.

Proposition 3. *$([\mathbb{N}]^{\aleph_0}, \tau)$ is a separable completely metrizable space.*

With this space in mind, we say that a set A is a **Borel set** if it is in the smallest σ-algebra containing the open sets. It is in this context where we find the condition our coloring has to obey to satisfy a theorem similar to Ramsey's.

Theorem 4 (Galvin-Prikry)**.** *Let*

$$[\mathbb{N}]^{\aleph_0} = P_0 \cup ... \cup P_{k-1}$$

be a partition where $P_0, P_1, ..., P_{k-1}$ are Borel sets. Then there exists a countable subset $A \subseteq \mathbb{N}$ such that $[A]^{\aleph_0} \subseteq P_r$ for some $r \in \{0, 1, ..., k-1\}$.

Ellentuck's topology

Ellentuck's approach begins with the introduction of another topology. Let $a \subseteq \mathbb{N}$ be finite and $A \subseteq \mathbb{N}$ be infinite such that $\max(a) < \min(A)$ (we denote this by $a < A$). We define

$$[a, A] := \{S \in [\mathbb{N}]^{\aleph_0} : a \subseteq S \subseteq a \cup A\}.$$

The family of sets $[a, A]$, for all $a < A$, forms the basis for what we shall call the **Ellentuck topology**.

Intuitively, we may think of the basic open sets $[a, A]$ in this topology as being of the following form:

$$\{\underbrace{n_1, n_2, n_3, ..., n_k}_{\text{all the elements of } a}, \underbrace{n_{k+j_1}, n_{k+j_2}, n_{k+j_3}, ...}_{\text{infinite elements of } A}\}.$$

An easy consequence of this definition is:

Lemma 5. *The Ellentuck topology is finer than the usual topology in $[\mathbb{N}]^{\aleph_0}$.*

In this environment it will be useful to have a notion of "almost open sets", we formalize this in what follows.

Definition 6. Let (X, τ) be a topological space, we say that a set $B \subseteq X$ has the **Baire property (BP)** if there is an open set $U \in \tau$ such that

$$B \Delta U := (B \setminus U) \cup (U \setminus B) \text{ is meager}$$

A general fact that follows is that the family of sets with the Baire property forms a particular σ-algebra.

Proposition 7. *Let (X, τ) be a topological space. Then the set of all subsets with the Baire property is the σ-algebra generated by the set of all open sets and meager sets.*

Ramsey's sets

Going back to the combinatorial aspects of our studies, we define two concepts that relate partitions to open sets. In these definitions it will be convenient to denote $\sim X := [\mathbb{N}]^{\aleph_0} \setminus X$.

Definition 8. Let X be a subset of $[\mathbb{N}]^{\aleph_0}$, we say that X is a **Ramsey set** if there is some set $[\emptyset, A]$ such that either

1. $[\emptyset, A] \subseteq X$, or
2. $[\emptyset, A] \subseteq \sim X$.

Definition 9. Let X be a subset of $[\mathbb{N}]^{\aleph_0}$, we say that X is a **completely Ramsey set** if, for any set $[a, A]$ there is some subset $B \subseteq A$ such that either

1. $[a, B] \subseteq X$, or
2. $[a, B] \subseteq \sim X$.

Proposition 10. *Every open set in the Ellentuck topology is completely Ramsey.*

Lemma 11. *Every nowhere dense set X in the Ellentuck topology is completely Ramsey.*

We can then extend this result to meager sets. In fact, this comes as a consequence of the fact that *in the Ellentuck topology a set is meager if, and only if, it is nowhere dense*.

Lemma 12. *Every meager set X in the Ellentuck topology is completely Ramsey.*

In fact, from Proposition 10 and Lemma 12 it follows that all sets with the Baire property are completely Ramsey.

Ellentuck's theorem

With what we developed in the last subsection, we have enough to present the main results. The first being that the implication discussed in the previous subsection is actually an equivalence.

Theorem 13 (Ellentuck)**.** *For any $X \subseteq [\mathbb{N}]^{\aleph_0}$, X is completely Ramsey if, and only if, X has the Baire property in the Ellentuck topology.*

The Galvin-Prikry theorem now follows easily:

Proof of Theorem 4. First, let us consider $k = 1$. Then $[\mathbb{N}]^{\aleph_0} = P_0 \cup P_1$ and $P_1 = \sim P_0$. Since P_0 is a Borel set, it is in the σ-algebra of open sets for the Ellentuck topology (Lemma 5). By Proposition 7, P_0 has the Baire property and by Theorem 13, it is completely Ramsey. Since $[\mathbb{N}]^{\aleph_0} = [\emptyset, \mathbb{N}]$, there is some infinite $B \subseteq \mathbb{N}$ such that either $[B]^{\aleph_0} = [\emptyset, B]$ is a subset of P_0 or P_1.

Now, suppose it is true for some $k \geq 1$, we will that it is also true for $k+1$: We may rewrite any given partition

$$[\mathbb{N}]^{\aleph_0} = P_0 \cup ... \cup P_{k-1} \cup P_k$$

as $[\mathbb{N}]^{\aleph_0} = Q \cup P_k$. As this is a 2-coloring, by what was previously proved, there is some infinite $B \subseteq \mathbb{N}$ such that either $[B]^{\aleph_0} \subseteq P_k$ (in which case we are done) or $[B]^{\aleph_0} \subseteq Q$ (where we get the k case, and are also done). □

Acknowledgments

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References

[1] T. J. Carlson and S. G. Simpson. *Topological Ramsey Theory.* In: Nešetřil, J., Rödl, V. (eds.) Mathematics of Ramsey Theory, pp. 172 – 183. John Wiley & Sons, 1990.

[2] E. Ellentuck. *A new proof that analytic sets are Ramsey*. Journal of Symbolic Logic, 1974.

[3] A. S. Kechris. *Classical Descriptive Set Theory*. Springer-Verlag, 1995.